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An Algorithm for Computing the Gamma C.D.F. to a Specified Accuracy

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The cumulative distribution function (c.d.f.) of the gamma distribution (also known as the incomplete gamma ratio) is

$$P(x,\alpha) = \int_0^x t^{\alpha-1}e^{-t}dt/\Gamma(\alpha) \quad (x>0;\alpha>0) \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$. [The left side of (1) is not standard statistical notation but is easier to work with in the derivations which follow]. The upper tail area of the distribution is

$$Q(x,\alpha) = \int_x^\infty t^{\alpha-1}e^{-t}dt/\Gamma(\alpha) \quad (x>0;\alpha>0).$$

$$= 1 - P(x,\alpha)$$

These functions are useful in computing the c.d.f.'s of the χ^2, noncentral χ^2, and Poisson distributions. For example,

$$\Pr\{\chi^2(\nu)<x\} = P(x/2,\nu/2) \quad (x>0;\nu>0) \text{ and}$$

$$\Pr\{Po(\lambda)<x\} = Q(\lambda,1+x) \quad (x=0,1,2,...;\lambda>0).$$

Two recent computer algorithms for computing $P(x,\alpha)$ are those of Bhattacharjee [2] and Lau [5]. The former uses a series expansion or continued fraction depending on the values of x and α. The latter uses a single series expansion for all x and α. In both cases the criterion for truncating the series is that the current term is less than a specified limit. As will be shown, this does not guarantee that the result will be accurate to that limit. Versions of these algorithms have been incorporated into software packages available at NBS [4,6,7].

In this note a modification of the Lau algorithm is presented which efficiently computes $P(x,\alpha)$ to a user-specified absolute accuracy. The algorithm makes use of the recurrence relation (see equation 6.5.21 of Abramowitz and Stegun [1])

$$P(x,\alpha) = x^\alpha e^{-x}/\Gamma(\alpha+1) + P(x,\alpha+1) \tag{2}$$

which relates the lower tail areas of the $\Gamma(\alpha)$ and $\Gamma(\alpha+1)$ distributions. If (2) is applied n times then

$$P(x,\alpha) = \sum_{i=1}^{n} x^{\alpha+i-1}e^{-x}/\Gamma(\alpha+i) + P(x,\alpha+n). \tag{3}$$

Given x, α, and an absolute accuracy limit $\varepsilon>0$, there will be a non-negative integer n such that $P(x,\alpha+n)<\varepsilon$ (the proof is left to the reader). The summation of n terms will then be an acceptable approximation to $P(x,\alpha)$.

An upper bound on $P(x,\alpha+n)$ is readily obtained by more applications of (2), hence

$$\begin{aligned} P(x,\alpha+n) &= \sum_{i=n+1}^{\infty} x^{\alpha+i-1}e^{-x}/\Gamma(\alpha+i) \\ &= \frac{x^{\alpha+n}e^{-x}}{\Gamma(\alpha+n+1)}\left[1 + \sum_{i=1}^{\infty} \frac{x^{i}}{(\alpha+n+1)(\alpha+n+2)\ldots(\alpha+n+i)}\right] \\ &< \frac{x^{\alpha+n}e^{-x}}{\Gamma(\alpha+n+1)}\left[1 + \sum_{i=1}^{\infty} \frac{x^{i}}{(\alpha+n+1)^{i}}\right] \\ &< \frac{x^{\alpha+n}e^{-x}}{\Gamma(\alpha+n+1)[1-x/(\alpha+n+1)]} \end{aligned} \tag{4}$$

when $x<\alpha+n+1$. When n becomes large enough so that the right hand side (RHS) of (4) is $<\varepsilon$, the series in (3) is truncated.

The above algorithm produces a sequence of gamma distributions whose lower tail areas steadily decrease when x is fixed, thus it can be called the "lower tail" method. When $x<\alpha$ relatively few terms may be required for convergence, but when $x>>\alpha$ an exceedingly large number of terms may be required. The Lau algorithm uses the above procedure for all x except that truncation occurs when $x^{\alpha+n}e^{-x}/\Gamma(\alpha+n+1) < \varepsilon$. This value is considerably smaller than the RHS of (4) when x is near (but less than) $\alpha+n+1$.

The next question to consider is the feasibility of applying the reverse procedure when $x>\alpha$, that is, producing a sequence of gamma distributions whose upper tail areas steadily decrease when x is fixed. This algorithm, called the "upper tail" method, makes use of the recurrence relation

$$Q(x,\alpha) = x^{\alpha-1}e^{-x}/\Gamma(\alpha) + Q(x,\alpha-1) \tag{5}$$

which is easily derived from (2). If (5) is applied m times then

$$Q(x,\alpha) = \sum_{i=1}^{m} x^{\alpha-i}e^{-x}/\Gamma(\alpha-i+1) + Q(x,\alpha-m) \tag{6}$$